Journal of Mathematical Sciences: Advances and Applications
Volume 1, Number 3, 2008, Pages 601-622

A WAVELET MULTIGRID METHOD APPLIED TO THE STOKES AND INCOMPRESSIBLE NAVIER-STOKES PROBLEMS

DOREEN DE LEON

Department of Mathematics
California State University, Fresno
5245 N. Backer Ave., M/S PB 108
Fresno, CA 93740-8001, U. S. A.
e-mail: doreendl@csufresno.edu

Abstract

The standard multigrid procedure performs poorly or may break down when used to solve certain problems. The wavelet multigrid method solves this problem by using a wavelet transform and Schur complements to determine the coarse grid and intergrid transfer operators. This method, previously applied to diffusion and advection-dominated advection-diffusion problems, here is modified for application to systems of partial differential equations, particularly the steady-state Stokes and incompressible Navier-Stokes equations. Numerical results demonstrate the effectiveness of the method applied to these problems.

1. Introduction

The multigrid method is very useful in increasing the efficiency of iterative methods used to solve systems of algebraic equations approximating partial differential equations. However, when confronted by certain problems, the standard multigrid procedure converges slowly, with a rate dependent on mesh size, or may even break down. In [12], a

2000 Mathematics Subject Classification: Primary 65N55; Secondary 35Q99.

Keywords and phrases: multigrid, wavelets.

Received October 03, 2008

wavelet multigrid method is described, and numerical results demonstrate its effiectiveness in solving diffusion problems with discontinuous and highly oscillatory coefficients of diffusion, as well as advection-dominated advection-diffusion problems. In this paper, the wavelet multigrid method is modified for application to systems of partial differential equations, in particular the steady-state Stokes and incompressible Navier-Stokes problems.

Much of the research that has been devoted to the numerical solution of the steady-state incompressible Navier-Stokes problem using multigrid methods involves the full approximation scheme (FAS). In the majority of the works, a finite element or finite volume based approach is taken for the discretization of the Navier-Stokes equations, using either staggered grids or non-staggered grids, and, in some cases, different choices of basis functions for the velocity and pressure terms. For example, in [1], a finite volume discretization is used, along with a FAS multigrid scheme, to solve numerically the incompressible Navier-Stokes equations written in conservative form. In [13], the Navier-Stokes equations are discretized on triangular grids and the control volume procedure is used (which involves integrating over a control volume to obtain the equations to be discretized). In that paper, FAS is also used to derive the coarse grid equations, and under-relaxation is used to prevent instability. The solution is obtained by solving the pressure equations (determined by satisfying the continuity equation) and then the momentum equations. In [18], a hybrid multigrid scheme is developed, which combines the approaches of multigrid methods (either linear or FAS), Newton-Krylov methods, and pressure-correction methods (SIMPLE and SIMPLER developed in [16, 17]).

R. Webster uses an algebraic multigrid approach to the solution of the incompressible Navier-Stokes equations in [26, 27]. In his work, equations are discretized using unstructured finite element meshes, employing different interpolations for pressure and velocity, but elements of equal order. The discrete equations are obtained by enforcing the conservation laws on the control volumes, and in [26] under-relaxation is used in the iterative scheme. In [27], the defect-correction method is used

to solve to second order accuracy, thus eliminating the need for under-relaxation.

In [21], Anton Schüller describes a finite difference approach to the solution of the Navier-Stokes equations which involves a re-statement of the problem in order to make the finite difference discretization more efficient and eliminate the need for the addition of an artificial stabilizing term. The standard five-point discretization for second order terms and centered differencing for the first order terms are used to discretize the resulting equations.

In other papers, multigrid methods are used as preconditioners for the Newton-Krylov method, e.g., [19, 23]. Both [19] and [23] use an inexact Newton's method to linearize the problem.

Due to the properties of wavelets, it seems natural to extend the wavelet multigrid method developed in [11, 12] to apply it to systems of linear partial differential equations, such as the steady-state Stokes equations, and to nonlinear problems, in particular to the steady-state incompressible Navier-Stokes equations.

The organization of this paper is as follows. In Section 2, some background on multigrid methods and wavelets is given. Section 3 describes the wavelet multigrid method developed in [12] and the necessary modifications for application to the Stokes and Navier-Stokes equations. In Section 4, numerical considerations are discussed. Section 5 presents some numerical results of applying the wavelet multigrid method to the Stokes equations and to the Navier-Stokes problem with different Reynold's numbers. The rapid convergence, independent of the Reynold's number, of the wavelet multigrid method is demonstrated for the Navier-Stokes problem.

2. Background

2.1. Multigrid methods

The problem we are concerned with solving is the system of linear equations

$$Au = b, \tag{1}$$

where A and b arise from discretization of a differential equation on some grid Ω^h, where h is the step size.

For notational purposes, we briefly describe the V-cycle method used in this paper. Given some interpolation operator, I_{2h}^h, where the superscript refers to the fine grid and the subscript refers to the coarse grid, and a restriction operator, I_h^{2h}, we can define a multigrid method recursively. For the two-level V-cycle method, first relax a few (usually one or two) steps on the fine grid Ω^h to get an initial guess u^h. Then, compute the residual $r^h = b^h - A^h u^h$; restrict the residual to the coarse grid Ω^{2h}, $r^{2h} = I_h^{2h} r^h$; and solve the error equation

$$A^{2h} e^{2h} = r^{2h}$$

on the coarse grid. Finally, set $u^h = u^h + I_{2h}^h e^{2h}$ and relax again a few (usually one or two) steps on the fine grid. Based on this two-level method, the V-cycle multigrid scheme is defined recursively. Some good references for multigrid methods are [7, 14, 25].

One type of multigrid scheme is algebraic multigrid, which uses only the structure of the matrix in the problem to determine the coarsening process (choice of coarse grid and definition of interpolation/restriction operators). This process is performed in order to ensure that the range of interpolation approximates the errors not sufficiently reduced via relaxation. For a more detailed description of algebraic methods, see, e.g., [20, 8, 15, 22, 25]. Note that in [8] the relation between algebraic multigrid and Schur complements is discussed. Algebraic multigrid methods are of particular interest, in that they are the nearest methods to the approach taken in this paper.

For certain problems, including elliptic problems with discontinuous or highly oscillatory coefficients, as well as advection-dominated problems, standard multigrid methods do not converge as expected. One difficulty is that the small eigenvalues of A are not necessarily associated with smooth eigenfunctions, a key assumption for the standard multigrid

method. For such problems, it is not as simple to approximate the smooth eigenfunctions on the coarse grids. Thus, new methods for restriction and interpolation, or for treating the entire problem, must be found.

2.2. Wavelets

For notational purposes, a brief description of wavelets follows. For more details, the reader is referred to [9, 10]. Wavelets basically separate data (or functions or operators) into different frequency components and analyze them by scaling. The wavelets can be chosen to form a complete orthonormal basis of $L^2(\mathbb{R})$. And, due to the scaling of the wavelet functions, they have time- or space-widths that are related to their frequency: at high frequencies, they are narrow, and at low frequencies, they are broader. Therefore, they provide good localization of functions in both the frequency domain and physical space, and representation by wavelets seems natural to apply to the analysis of fine and coarse scales.

A multiresolution analysis (MRA) consists of a sequence of closed subspaces $V_j \subset V_{j-1}$ of $L^2(\mathbb{R})$, the scaling spaces, that satisfy certain conditions. For every $j \in \mathbb{Z}$, W_j is defined as the orthogonal complement of V_j in V_{j-1}. W_j are called the *wavelet spaces*. Define H_j and G_j to be the operators that transform the basis of the space V_j to the bases of the spaces V_{j+1} and W_{j+1}, respectively. The properties of H_j, the scaling operator, and G_j, the wavelet operator, are (assuming H_j and G_j are real-valued)

(i) $H_j^T H_j + G_j^T G_j = I$,

(ii) $H_j G_j^T = G_j H_j^T = 0$,

(iii) $H_j H_j^T = G_j G_j^T = I$.

H_j and G_j can be thought of in terms of filter theory, with H_j being a low-pass filter (i.e., allowing only low frequency values to pass) and G_j being a high-pass filter (i.e., allowing only high frequency values to pass).

The wavelet transform, $\mathcal{W}_j : V_j \to V_{j+1} \oplus W_{j+1}$, is defined by

$$\mathcal{W}_j = \begin{pmatrix} H_j \\ G_j \end{pmatrix}.$$

Note that $\mathcal{W}_j$ is orthogonal due to the properties of H_j and G_j.

In the discrete context, the wavelet operators are computationally effcient. With respect to the Haar multiresolution analysis, application of the low-frequency operator (H_j) to a vector in $\mathbb{R}^n$ involves only $2n$ operations. The same holds for the high-frequency operator (G_j). So, the application of the wavelet transform requires only $4n$ operations. In general, application of the wavelet transform requires $\mathcal{O}(n)$ operations, assuming a finite number of coefficients for the low- and high-frequency operators.

In two dimensions, the tensor product of one-dimensional multiresolution analyses is used. So $\mathbf{V}_j$ is defined by $\mathbf{V}_j = V_j \otimes V_j$. These spaces $\mathbf{V}_j$ then form an MRA in $L^2(\mathbb{R}^2)$. For each $j \in \mathbb{Z}$, $\mathbf{W}_j$ is defined as the orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j-1}$. So,

$$\begin{aligned} \mathbf{V}_{j-1} &= \mathbf{V}_{j-1} \otimes \mathbf{V}_{j-1} \\ &= (V_j \otimes V_j) \oplus [(W_j \otimes V_j) \oplus (V_j \otimes W_j) \oplus (W_j \otimes W_j)] \\ &= \mathbf{V}_j \oplus \mathbf{W}_j. \end{aligned} \tag{2}$$

Then, analogous to the one dimensional case, define the operators H_j and G_j so that

$$H_j = H_j^y H_j^x$$

and

$$G_j = \begin{pmatrix} G_j^y H_j^x \\ H_j^y G_j^x \\ G_j^y G_j^x \end{pmatrix}.$$

H_j and G_j have the same properties as their one-dimensional analogues.

As in the one-dimensional case, the wavelet transform, $\mathcal{W}_j : \mathbf{V}_j \to \mathbf{V}_{j+1} \oplus \mathbf{W}_{j+1}$ is defined by

$$\mathcal{W}_j = \begin{pmatrix} H_j \\ G_j \end{pmatrix}.$$

Again, due to the properties of H_j and G_j, $\mathcal{W}_j$ is orthogonal.

3. The Wavelet Multigrid Method

3.1. A brief description of the method

The following is a brief description of the wavelet multigrid method, with full details to be found in [11, 12].

Given the problem

$$L_j V = F, \tag{3}$$

where L_j represents the operator obtained by discretizing a two-dimensional partial differential equation on the fine grid, apply the wavelet transform to both sides of the equation and use the orthogonality of $\mathcal{W}_j$ to obtain

$$(\mathcal{W}_j L_j \mathcal{W}_j^T)\mathcal{W}_j V = \mathcal{W}_j F$$

$$\Rightarrow (\mathcal{W}_j L_j \mathcal{W}_j^T) \begin{pmatrix} V_L \\ V_{LH} \\ V_{HL} \\ V_{HH} \end{pmatrix} = \begin{pmatrix} F_L \\ F_{LH} \\ F_{HL} \\ F_{HH} \end{pmatrix}, \tag{4}$$

where $V_L, F_L \in \mathbf{V}_j$ and $(V_{LH}, V_{HL}, V_{HH})^T, (F_{LH}, F_{HL}, F_{HH})^T \in \mathbf{W}_j$. The subscripts L and H derive from the fact mentioned in Section 2.2 that the operator H_j can be viewed as a low-pass filter and G_j can be viewed as a high-pass filter.

$\mathcal{W}_j L_j \mathcal{W}_j^T$ is computed, and the resulting matrix, denoted by $\widetilde{L}_j$, is partitioned to obtain

$$\widetilde{L}_j = \mathcal{W}_j L_j \mathcal{W}_j^T = \begin{pmatrix} T_j & B_j \\ C_j & D_j \end{pmatrix}. \tag{5}$$

Then, the block UDL decomposition of $\widetilde{L}_j$, where U is block upper triangular with unit diagonal, D is block diagonal, and L is block lower triangular with unit diagonal, is computed and is used to find $\widetilde{L}_j^{-1}$. Define

$$V = \begin{pmatrix} V_L \\ V_H \end{pmatrix} = \begin{pmatrix} V_L \\ V_{LH} \\ V_{HL} \\ V_{HH} \end{pmatrix}$$

and similarly $F = \begin{pmatrix} F_L \\ F_H \end{pmatrix}$. Solving for $V = \begin{pmatrix} V_L \\ V_H \end{pmatrix}$ gives

$$V = \begin{pmatrix} V_L \\ V_H \end{pmatrix} = \begin{pmatrix} (T_j - B_j D_j^{-1} C_j)^{-1}(H_j - B_j D_j^{-1} G_j) \\ -D_j^{-1} C_j (T_j - B_j D_j^{-1} C_j)^{-1}(H_j - B_j D_j^{-1} G_j) + D_j^{-1} G_j \end{pmatrix} F.$$

So,

$$V = (H_j^T - G_j^T D_j^{-1} C_j \quad G_j^T) \begin{pmatrix} (T_j - B_j D_j^{-1} C_j)^{-1} & 0 \\ 0 & D_j^{-1} \end{pmatrix} \begin{pmatrix} H_j - B_j D_j^{-1} G_j \\ G_j \end{pmatrix} F. \tag{6}$$

Denote

$$I_{2h}^h = \sqrt{2}(H_j^T - G_j^T D_j^{-1} C_j) \tag{7}$$

and

$$I_h^{2h} = \frac{\sqrt{2}}{2}(H_j - B_j D_j^{-1} G_j), \tag{8}$$

as the interpolation and restriction operators, respectively. Substituting the interpolation and restriction operators defined in (7) and (8) into (6) gives

$$V = I_{2h}^{h}(T_j - B_j D_j^{-1} C_j)^{-1} I_h^{2h} F + G_j^T D_j^{-1} G_j F. \tag{9}$$

In multigrid, the error correction on the coarse grid is sought; i.e., the equation being solved is (9) with V replaced by the error, e, and F replaced by the residual, r:

$$e = I_{2h}^{h}(T_j - B_j D_j^{-1} C_j)^{-1} I_h^{2h} r + G_j^T D_j^{-1} G_j r.$$

Assuming that $G_j^T D_j^{-1} G_j r$ is small, i.e., r is almost in Range(H_j^T), the error can be approximated by

$$e = I_{2h}^{h}(T_j - B_j D_j^{-1} C_j)^{-1} I_h^{2h} r,$$

resulting in

$$(T_j - B_j D_j^{-1} C_j) e^{2h} = I_h^{2h} h.$$

The above assumption is good for most of the classical iterative methods, like Jacobi and Gauss-Seidel. Therefore, the coarse grid operator is defined by

$$L_{j+1} = T_j - B_j D_j^{-1} C_j, \tag{10}$$

which is the Schur complement of D_j in $\widetilde{L}_j$.

The above procedure is repeatedly applied until the desired coarseness is reached.

3.2. The wavelet multigrid method applied to the steady-state Stokes and incompressible Navier-Stokes equations

Here, we consider the Stokes equations, given by

$$-\Delta u + p_x = f^u \text{ in } \Omega, \tag{11}$$

$$-\Delta v + p_y = f^v \text{ in } \Omega, \tag{12}$$

$$u_x + v_y = 0 \text{ in } \overline{\Omega}, \tag{13}$$

and the incompressible Navier-Stokes equations, given by

$$-\Delta u + \mathrm{Re}(uu_x + vu_y + p_x) = f^u \text{ in } \Omega, \tag{14}$$

$$-\Delta v + \mathrm{Re}(uv_x + vv_y + p_y) = f^v \text{ in } \Omega, \tag{15}$$

$$u_x + v_y = 0 \text{ in } \overline{\Omega}. \tag{16}$$

The Stokes and incompressible Navier-Stokes problems have three unknowns, u, v, and p. This necessitates a modification of the wavelet multigrid method. Let H_j^u be the scaling operator that will be applied to the discrete values for u. Similarly, define H_j^v and H_j^p. The wavelet operators G_j^u, G_j^v, and G_j^p are defined in the same way. Then, let

$$H_j = \begin{pmatrix} H_j^u & 0 & 0 \\ 0 & H_j^v & 0 \\ 0 & 0 & H_j^p \end{pmatrix}.$$

Define G_j in the same manner. The operators H_j and G_j thus defined satisfy the properties (i)-(iii) of the scaling and wavelet operators defined in Subsection 3.1. The wavelet transform $\mathcal{W}_j$ is defined by

$$\mathcal{W}_j = \begin{pmatrix} H_j \\ G_j \end{pmatrix}.$$

$\mathcal{W}_j$ is orthogonal due to the properties of H_j and G_j.

The wavelet multigrid method then follows as in Subsection 3.1, letting

$$V = \begin{pmatrix} u \\ v \\ p \end{pmatrix}.$$

Thus, application of $\mathcal{W}_j$ to V, which is given by

$$\mathcal{W}_j V = \begin{pmatrix} H_j V \\ G_j V \end{pmatrix},$$

takes the projection of u, v, and p onto the scaling space (or low-frequency

space) as a result of the operations of H_j^u, H_j^v, and H_j^p on u, v, and p, respectively, and stores those values in the upper half of the resulting vector; and it takes the projection of u, v, and p onto the wavelet space (or high-frequency space) through the operations of G_j^u, G_j^v, and G_j^p on u, v, and p, respectively, and stores those values in the bottom half of the resulting vector.

4. Numerical Considerations

Although D_j is not dense, its inverse is dense due to fill-in.

One step towards increasing the efficiency of the algorithm in this area is to use an approximate inverse. In this work, a factorized sparse approximate inverse based on biconjugation, AINV, is used (see, e.g., [2, 3, 4, 5, 28]). The goal of AINV is to compute an approximate inverse of D_j, $ZD^{-1}W^T$, where Z and W are upper triangular matrices with unit diagonals whose columns are D_j-biconjugate, and D is diagonal. The algorithm used to compute the matrix components can be interpreted as a two-sided Gramm-Schmidt orthogonalization process with respect to the bilinear form associated with the matrix to be inverted, D_j. Fill-in in the inverse is reduced by using drop tolerances to eliminate "small" fill-in in the updates to Z and W. In addition, after $ZD^{-1}W^T$ is computed, fill-in is further reduced by applying drop tolerances to eliminate "small" entries in the resulting approximate inverse.

4.1. Cost of computing the approximate inverse

The bulk of the cost of the AINV algorithm is incurred in computing Z and W and can be measured in terms of sparse matrix-sparse vector products and sparse vector-sparse vector products. The construction of Z and W requires n iterations of the algorithm, where n is the number of rows of the matrix whose inverse is being computed. Each iteration requires two sparse vector-sparse matrix products with a vector that is of length n and a matrix that is $n \times (n - i + 1)$, where i ranges from 1 to n, and two sparse vector-sparse vector products, in which one vector is a

column vector of size n and one is a row vector of size $n - i$. Thus, a total of $2n$ sparse matrix-sparse vector products and $2n$ sparse vector-sparse vector products must be computed.

The final major contribution to the cost of computing the approximate inverse is the computation of $ZD^{-1}W^T$, which requires nnz scalar products, where nnz is the number of nonzero elements in Z, followed by one sparse matrix-sparse matrix product.

The specific values of the drop tolerance follow. For both the Stokes and the Navier-Stokes problems, the drop tolerance for Z ranges from 10^{-6} to 10^{-8} and the drop tolerance for W ranges from 10^{-6} to 10^{-7}, depending on the fine grid size and on the level. After the approximate inverse is computed, further dropping is done by using a drop tolerance which is computed by multiplying the minimum value on the diagonal of the approximate D_j^{-1} by a specified tolerance. For the Stokes equations, this tolerance is either 10^{-4} or 10^{-6}, depending on the fine grid size and on the level. For the Navier-Stokes problem, this tolerance is 10^{-4}, 10^{-5}, or 10^{-6}, depending on the Reynold's number, the fine grid size, and the level.

4.2. Storage and other computational issues

The bulk of the remaining computational work occurs in the construction of the intergrid transfer and coarse grid operators. Given a problem with $N = n^2$ grid points, the application of the wavelet transform requires $\mathcal{O}(N)$ operations (see [6] for more details regarding the fast wavelet transform). The construction of the intergrid transfer and coarse grid operators each requires two sparse matrix-sparse matrix products and one sparse matrix-sparse matrix difference.

The storage requirements of the coarse grid and intergrid transfer operators are minimized by using sparse matrix storage techniques, resulting in storage requirements of the order of the number of nonzero elements in each matrix.

5. Numerical Results

5.1. The Stokes equations

As is done in [21], differentiating (11) with respect to x, differentiating (12) with respect to y, summing the resulting equations, and using the continuity condition (13), we obtain the following system of equations:

$$-\Delta u + p_x = f^u \text{ in } \Omega, \tag{17}$$

$$-\Delta v + p_y = f^v \text{ in } \Omega, \tag{18}$$

$$\Delta p = f_x^u + f_y^v \text{ in } \Omega, \tag{19}$$

$$u_x + v_y = 0 \text{ on } \partial\Omega. \tag{20}$$

The new pressure equation (19) requires a boundary condition. The boundary condition for p is determined from (17) and (18) to be

$$\frac{\partial p}{\partial n} = (\Delta \mathbf{u} + \mathbf{f}) \cdot \mathbf{n}. \tag{21}$$

The equations (17)-(19), (21) are discretized using centered differencing for the first order pressure terms (p_x, p_y) and the usual five-point discretization for the second order terms $(\Delta u, \Delta v, \Delta p)$. Forward differencing (left-hand side and bottom of the square) and backward differencing (right-hand side and top of the square) are used to discretize the continuity equation (20). The discretized continuity equation is used to obtain the outermost interior values on the left and right boundaries of the square (for u) and on the top and bottom (for v). This can easily be done because u and v are specified on the boundary of the unit square. The corner values of p do not appear directly in the discretization and are assigned the value of the average of the adjacent boundary values for p.

Both f^u and f^v are set to zero for the numerical calculations and Ω is the unit square centered at $(0.5, 0.5)$. The V-cycle multigrid algorithm is used, with Gauss-Seidel as the smoother, but fifty iterations are performed on the coarsest grid in lieu of an exact solve since the matrix is

singular. Fewer iterations are necessary on the coarsest grid if it is sufficiently coarse.

The wavelet multigrid method is applied to the Stokes equations with three different sets of boundary conditions. In all problems, the stopping criterion is $\| r_k \|_2 < 10^{-5}$, where r_k is the residual obtained from the kth iteration of the method, and the initial guess for all problems is the ones vector. The three boundary conditions used are

$$\begin{cases} u = 0, \\ v = 0; \end{cases}$$

$$\begin{cases} u = \begin{cases} 16x^2(1-x)^2, & \text{if } y = 1, \\ 0, & \text{otherwise,} \end{cases} \\ v = 0; \end{cases} \tag{22}$$

and

$$\begin{cases} u = \begin{cases} 1, & \text{if } y = 1, \\ 0, & \text{otherwise,} \end{cases} \\ v = 0. \end{cases} \tag{23}$$

Tables 1, 2, and 3 display the average convergence factor per cycle of the wavelet multigrid method using Haar wavelets and the number of iterations to convergence for each of the three boundary conditions. These results are compared to results for the W-cycle multigrid method (since the V-cycle method using these operators does not converge), using nine-point interpolation, full-weighting restriction (a constant multiple of the adjoint of the nine-point interpolation, scaled so that the sum of the weights is one (see [7] and [14])), and a coarse grid operator defined by discretizing the equation on a grid with the appropriate step size. The W-cycle is another type of multigrid method, in which a return to the fine grid size is accomplished in stages: the problem is transferred to the coarsest grid, then interpolated up to the next coarsest grid, restricted to the coarsest grid, interpolated up two levels, restricted to the coarsest grid, and so on, until the finest grid is reached. With respect to the W-cycle multigrid an odd number of gridpoints is required, so that we have a 15×15 grid, a 31×31 grid, and a 63×63 grid in the interior. The tables

demonstrate results with a fixed coarsest grid size of 4×4 for each fine mesh size. These results show that the convergence rate of the wavelet multigrid method is essentially independent of the fine mesh size.

Table 1. Average convergence factor per cycle and number of iterations to convergence for the Stokes equations with homogeneous boundary conditions

Wavelet Multigrid Method			*W*-cycle Multigrid Method	
fine grid size	avg. conv./cycle	iters	avg. conv./cycle	iters
16×16 (3 levels)	0.3801	12	0.9683	330
32×32 (4 levels)	0.4954	15	0.9930	1498
64×64 (5 levels)	0.5346	16	-	> 2000

Table 2. Average convergence factor per cycle and number of iterations to convergence for the Stokes equations with the boundary conditions in (22)

Wavelet Multigrid Method			*W*-cycle Multigrid Method	
fine grid size	avg. conv./cycle	iters	avg. conv./cycle	iters
16×16 (3 levels)	0.3734	12	0.9684	329
32×32 (4 levels)	0.4847	15	0.9929	1489
64×64 (5 levels)	0.5271	16	-	> 2000

5.2. The incompressible Navier-Stokes equations

The Navier-Stokes equations may be rewritten (in a similar way as the Stokes equations) in a form that will allow the solution algorithm to be more efficient and eliminate the need for the addition of an artificial stabilizing term. This theorem was also stated in a slightly different form in [21] and [25].

Theorem 1. *Given* Ω, *a bounded domain in* $\mathbb{R}^2$, *and functions* $u(x, y)$ *and* $v(x, y)$ *in* $C^3(\Omega) \cap C^1(\overline{\Omega})$, $p(x, y)$ *in* $C^2(\Omega)$, *and* f^u, $f^v \in C^1(\Omega)$, *the system of equations* (14) - (16) *is equivalent to the system*

Table 3. Average convergence factor per cycle and number of iterations to convergence for the Stokes equations with the boundary conditions in (23)

Wavelet Multigrid Method			*W*-cycle Multigrid Method	
fine grid size	avg. conv./cycle	iters	avg. conv./cycle	iters
16×16 (3 levels)	0.3829	12	0.9684	329
32×32 (4 levels)	0.4940	15	0.9930	1492
64×64 (5 levels)	0.5361	16	-	> 2000

$$-\Delta u + \mathrm{Re}(uu_x + vu_y + p_x) = f^u \text{ in } \Omega, \tag{24}$$

$$-\Delta v + \mathrm{Re}(uv_x + vv_y + p_y) = f^v \text{ in } \Omega, \tag{25}$$

$$\Delta p + 2(v_x u_y - u_x v_y) = \frac{1}{\mathrm{Re}}(f^u_x + f^v_y) \text{ in } \Omega, \tag{26}$$

$$u_x + v_y = 0 \text{ on } \partial\Omega. \tag{27}$$

Proof. Differentiating equation (14) with respect to x and applying the continuity equation, equation (16), yields

$$-\Delta u_x + \mathrm{Re}(-u_x v_y + v_x u_y + uu_{xx} + vu_{yx} + p_{xx}) = f^u_x.$$

Similarly, differentiating equation (15) with respect to y and applying (16) yields

$$-\Delta v_y + \mathrm{Re}(u_y v_x - v_y u_x + uv_{xy} + vv_{yy} + p_{yy}) = f^v_y.$$

Adding these two equations and noting that $u_{yx} = u_{xy}$, we obtain

$$-\Delta(u_x + v_y) + 2\,\mathrm{Re}(u_y v_x - u_x v_y) + \mathrm{Re}\,\Delta p$$

$$+ \mathrm{Re}(u(u_x + v_y)_x + v(u_x + v_y)_y) = f^u_x + f^v_y.$$

Applying the continuity condition (16) to this sum and dividing by Re yields (26).

The new pressure equation (26) requires a boundary condition. The boundary condition for p is determined from (24) and (25) to be

$$\frac{\partial p}{\partial n} = \frac{1}{\mathrm{Re}}(\Delta \mathbf{u} - \mathrm{Re}(\mathbf{u} \cdot \nabla \mathbf{u}) + \mathbf{f}) \cdot \mathbf{n}. \tag{28}$$

Equations (24)-(28) are discretized using the standard five-point discretization for the second order terms. The advection terms (u_x, u_y, v_x, v_y) are discretized using first order upwind differencing in (24), (25), and (28). Centered differencing is used for the first order terms in (26) and for the first order pressure terms in (24), (25), and (28). Forward differencing (left-hand side and bottom of the square) and backward differencing (right-hand side and top of the square) are used to discretize the continuity equation (27). The equations for u and v at the outermost interior gridpoints are obtained from the discretized continuity equation. The corner values of p do not appear directly in the discretization and are assigned the value of the average of the adjacent boundary values for p.

Newton's method is used to linearize the equations. The basic procedure is as follows: let F_{NS}^{u} denote the operator arising from discretizing (24), F_{NS}^{v} denote the operator arising from discretizing (25), and F_{NS}^{p} denote the operator arising from discretizing (26). In each of these equations, substitute $u + \tilde{u}$, $v + \tilde{v}$, and $p + \tilde{p}$ for u, v, and p. Then, expand, delete any terms involving products of $\tilde{u}$ and $\tilde{v}$, and put all remaining terms not involving $\tilde{u}$, $\tilde{v}$, and $\tilde{p}$ on the right-hand side. This gives the approximation

$$(F_{NS}^{u})'(\tilde{u}, \tilde{v}, \tilde{p}) \approx f - F_{NS}^{u}(u. v, p)$$

for (24), and similarly for (25) and (26). The same is done for the continuity equation (27) and the boundary conditions. Then, writing the resulting system in matrix form yields

$$F'_{NS}\tilde{U}^{n+1} = f - F_{NS}U^n, \tag{29}$$

where F_{NS} is the matrix determined by the discretization of (24)-(28) and the boundary conditions for u and v. After solving for

$$\tilde{U}^{n+1} = \begin{pmatrix} \tilde{u}^{n+1} \\ \tilde{v}^{n+1} \\ \tilde{p}^{n+1} \end{pmatrix},$$

correct to obtain the improved solution

$$U^{n+1} = U^n + \tilde{U}^{n+1}.$$

The linear system of equations in (29) is solved using the wavelet multigrid method, defining L_j as the operator F'_{NS}, V as $\tilde{U}^{n+1}$, and F as the right-hand side, $f - F_{NS}U^n$.

Both f^u and f^v are set to zero for the numerical calculations and Ω is the unit square centered at $(0.5, 0.5)$. The wavelet multigrid method is applied to the Navier-Stokes equations with Reynold's numbers ranging from 10 to 200. The V-cycle multigrid algorithm is used, with Gauss-Seidel as the smoother, but fifty iterations are performed on the coarsest grid in lieu of an exact solve since the matrix is singular. Fewer iterations are necessary on the coarse grid if it is sufficiently coarsest. In all problems, the stopping criterion is $\| r_k \|_2 < 10^{-5}$, where r_k is the residual obtained from the kth iteration of the method, and the initial guess for all problems is the ones vector.

Tables 4 through 8 display the average convergence factor per cycle of the wavelet multigrid method using Haar wavelets and the number of iterations to convergence of the linearized problem for Reynold's numbers of 10, 50, 100, 150, and 200. The tables demonstrate results with a fixed coarsest grid size of 4×4 for each fine mesh size. These results show that the convergence rate of the wavelet multigrid method is essentially independent of both the fine mesh size and the Reynold's number.

Table 4. Average convergence factor per cycle and number of iterations to convergence for the linearized system in (29) with Reynold's number 10

fine grid size	avge. convergence/cycle	number of iterations
16×16 (3 levels)	0.3542	11
32×32 (4 levels)	0.3829	13
64×64 (5 levels)	0.4300	13

Table 5. Average convergence factor per cycle and number of iterations to convergence for the linearized system in (29) with Reynold's number 50

fine grid size	avge. convergence/cycle	number of iterations
16×16 (3 levels)	0.3854	12
32×32 (4 levels)	0.4031	13
64×64 (5 levels)	0.4700	15

Table 6. Average convergence factor per cycle and number of iterations to convergence for the linearized system in (29) with Reynold's number 100

fine grid size	avge. convergence/cycle	number of iterations
16×16 (3 levels)	0.3863	13
32×32 (4 levels)	0.4491	15
64×64 (5 levels)	0.5224	17

Table 7. Average convergence factor per cycle and number of iterations to convergence for the linearized system in (29) with Reynold's number 150

fine grid size	avge. convergence/cycle	number of iterations
16×16 (3 levels)	0.4271	14
32×32 (4 levels)	0.4621	15
64×64 (5 levels)	0.5438	19

Table 8. Average convergence factor per cycle and number of iterations to convergence for the linearized system in (29) with Reynold's number 200

fine grid size	avge. convergence/cycle	number of iterations
16×16 (3 levels)	0.4097	13
32×32 (4 levels)	0.4672	16
64×64 (5 levels)	0.5454	19

6. Conclusions

The wavelet multigrid method has been shown to be effective in solving the Stokes equations, a linear system of partial differential equations. In addition, it has been demonstrated that the wavelet multigrid method is effective in solving the incompressible Navier-Stokes problem for moderate Reynold's numbers. Also, the properties of wavelets should permit the wavelet multigrid method to be applied efficiently through the use of compression. The results shown in this paper have demonstrated that it is worthwhile to further explore the usefulness of the wavelet multigrid method in solving fluid dynamics problems.

References

[1] C. Becker, J. H. Ferziger, M. Peric and G. Scheuerer, Finite volume multigrid solutions of the two-dimensional incompressible Navier-Stokes equations, Robust Multigrid Methods, Notes on Numerical Fluid Mechanics, Vol. 23, pp. 37-47 Vieweg, Braunschweig, 1988.

[2] M. Benzi and M. Tuma, A sparse approximate inverse preconditioner for nonsymmetric linear systems, SIAM J. Sci. Comput. 19(3) (1998), 968-994.

[3] M. Benzi and M. Tuma, A comparative study of sparse approximate inverse preconditioners, Appl. Numer. Math. 30 (1999), 305-340.

[4] M. Benzi, Preconditioning techniques for large linear systems: a survey, J. Comput. Phys. 182 (2002), 418-477.

[5] M. Bollhöffer and Y. Saad, On the relations between ILUs and factored approximate inverses, SIAM J. Matrix Anal. Appl. 24(1) (2002), 219-237.

[6] G. Beylkin, R. Coifman, and V. Rokhlin, Fast wavelet transforms and numerical algorithms I, Comm. Pure Appl Math. 44 (1991), 141-183.

[7] W. L. Briggs, S. F. McCormick, and V. E. Henson, A Multigrid Tutorial, Second Edition, SIAM, Philadelphia, PA, 2000.

[8] W. Dahmen and L. Elsner, Algebraic multigrid methods and the Schur complement, Robust Multi-Grid Methods (Kiel, 1988), Notes on Numerical Fluid Mechanics, Vol. 23, pp. 58-68, Vieweg, Braunschweig, 1989.

[9] I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math., XLI, (7) (1988), 909-996.

[10] I. Daubechies, Ten Lectures on Wavelets, volume 61 of CBMS-NSF Series in Applied Mathematics SIAM, Philadelphia, PA, 1992.

[11] D. De Leon, Wavelet operators applied to multigrid methods (Ph. D. thesis), UCLA Mathematics Department CAM Report 00-22, June 2000.

[12] D. De Leon, A new wavelet multigrid method, J. Comput. Appl. Math. 220 (2008), 674-685.

[13] R. Jyotsna and S. P Vanka, A pressure based multigrid procedure for the Navier-Stokes equations on unstructured grids, Proceedings of the Seventh Copper Mountain Conference on Multigrid Methods, 1995.

[14] S. McCormick, Multigrid Methods, Frontiers in Applied Mathematics, SIAM, Philadelphia, PA, 1987.

[15] J. D. Moulton, J. E. Dendy Jr. and J. M. Hyman. The black box multigrid numerical homogenization algorithm. J. Comput. Phys. 142 (1998), 80-108.

[16] S. V. Patankar and D. B. Spalding, A calculation procedure for heat, mass and momentum transfer in three-dimensional parabolic flows Internat. J. Heat Mass Transfer 15 (1972), 1787-1806.

[17] S. V. Patankar, A calculation procedure for two-dimensional elliptic situations, Numer. Heat Transfer 4 (1981), 409-425.

[18] M. Pernice, A hybrid multigrid method for the steady-state incompressible Navier-Stokes equations, Electron. Trans. Numer. Anal. 10 (2000), 74-91.

[19] M. Pernice and M. D. Tocci, A multigrid-preconditioned Newton-Krylov method for the incompressible Navier-Stokes equations, SIAM J. Sci. Comput. 23(2) (2001), 398-418.

[20] J. W. Ruge and K. Stüben, Algebraic multigrid, Multigrid Methods, Frontiers in Applied Mathematics, SIAM, Philadelphia, PA, 1987, pp. 73-130.

[21] A. Schüller, A multigrid algorithm for the incompressible Navier-Stokes equations, Numerical Methods for Advection-Diffusion Problems (Kiel, 1989), volume 30 of Notes on Numerical Fluid Mechanics, pp. 124-133, Vieweg, Braunschweig, 1990.

[22] K. Stüben, Algebraic multigrid (AMG): an introduction with applications, Technical Report 70, GMD, November 1999.

[23] Syamsudhuha and D. J. Silvester, Efficient solution of the steady state Navier-Stokes equations using a multigrid preconditioned Newton-Krylov method, Int. J. Numer. Meth. Fluids 43 (2003), 1407-1427.

[24] Roger Temam, Navier-Stokes Equations: Theory and Numerical Analysis, 3rd ed., Studies in Mathematics and its Applications, Vol. 2, North-Holland, New York, 1984.

[25] U. Trottenberg, C. W. Oosterlee and A. Schüller, Multigrid, Academic Press, London, 2001.

[26] R. Webster, An algebraic multigrid solver for Navier-Stokes problems, Int. J. Numer. Meth. Fluids 18(8) (1994), 761-780.

[27] R. Webster, An algebraic multigrid solver for Navier-Stokes problems in the discrete second order approximation, Int. J. Numer. Meth. Fluids 22(11) (1996), 1103-1123.

[28] A. Yu. Yeremin and A. A. Nikishin, Factorized-sparse-approximate inverse preconditionings of linear systems with unsymmetric matrices, J. Math. Sci. 121(4) (2004), 2248-2257.

■